Synchronizing Automata and The Černý Conjecture

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Graduate Algebra Seminar
University of Colorado

April 11, 2013

These slides and other resources are available at

http://williamdemeo.wordpress.com

Synchronizing automata

A ***finite automaton*** is a triple $\mathbf{A} = \langle Q, \Sigma, \delta \rangle$ where

- Q is a set of *states*
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Let Σ^* denote the free monoid obtained by composing letters.

A *word* $w \in \Sigma^*$ is just a string of letters, $w = a_0 a_1 \ldots a_{n-1}$, which acts on a state $q \in Q$ as you expect:

$$wq = a_0 a_1 \ldots a_{n-1} q$$

SYNCHRONIZING AUTOMATA

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Example

$$Q = \{0, 1, 2, 3\}$$
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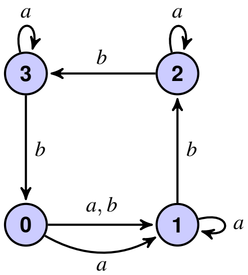

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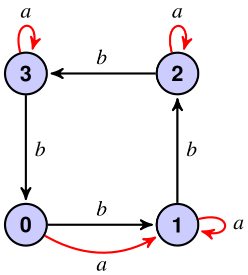

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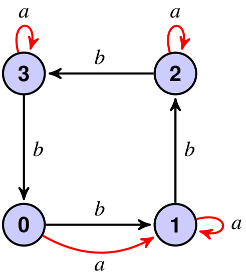

reset word:

abbbabbba

The notion was formalized in 1964 in a paper by Jan Černý, though implicitly it had been around since at least 1956.

The idea of synchronization is natural and of obvious importance: we aim to restore control over a device whose current state is unknown.

For example, our view of an orbiting satellite may be temporarily obstructed by the Moon; once it comes back into view, we regain control and reorient it (Černý's original motivation).

Synchronizing automata

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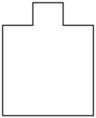
Suppose that one of the parts of a certain device has the following shape:



Such parts will move along a conveyor belt and must be sorted and oriented before assembly.

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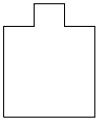

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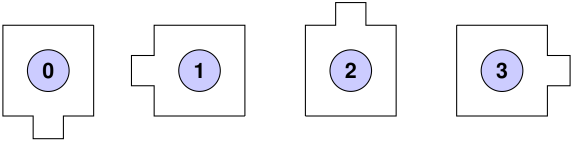
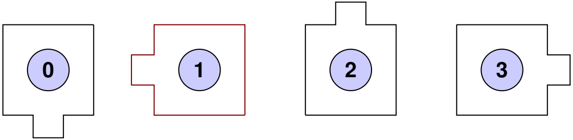
Assume that only four initial orientations are possible, namely,



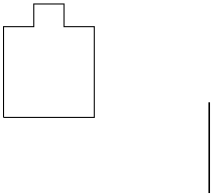


Prior to assembly the part must be in position **1**, *"bump-left"* orientation.

Problem: construct an orienter that will put the part in bump-left position independently of its initial orientation.

Synchronizing automata

Solution: position two types of obstacles, short and tall, in the path of the part. When the part passes a tall obstacle, it always rotates 90°. When it passes a short obstacle, it rotates by 90° iff it is in bump-down orientation.



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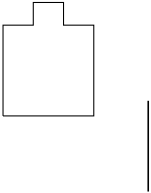

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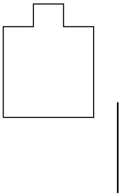

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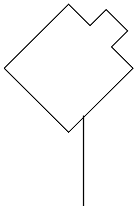

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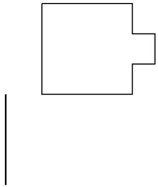

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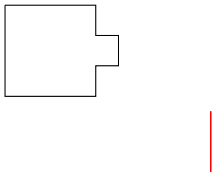

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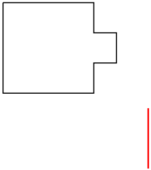

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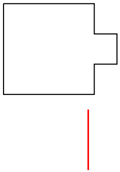

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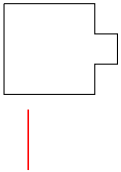

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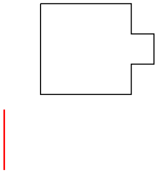

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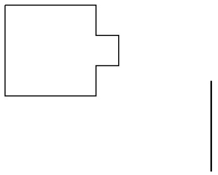

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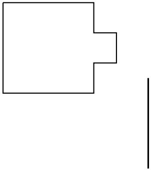

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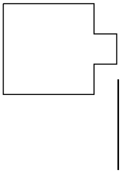

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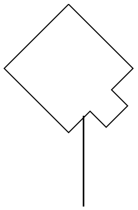

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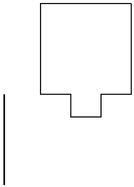

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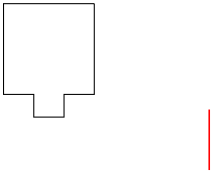

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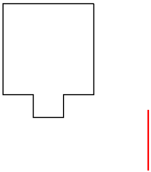

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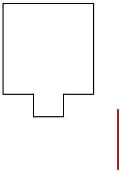

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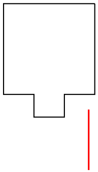

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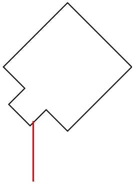

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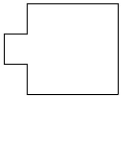

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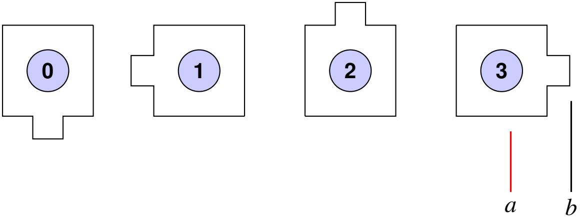
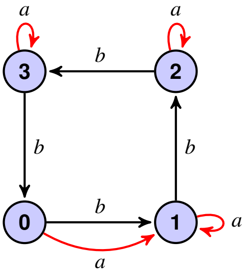
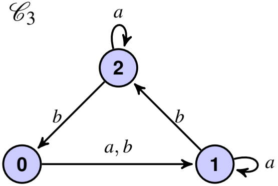




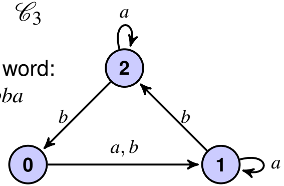
Černý's infinite family



Černý's infinite family

$\mathscr{C}_3$

reset word:
abba



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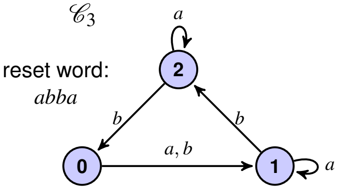

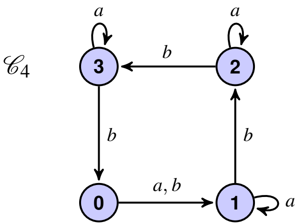

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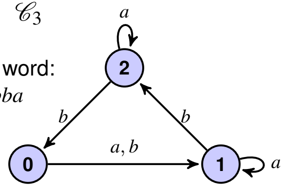
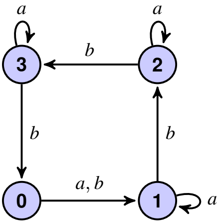


$\mathscr{C}_4$

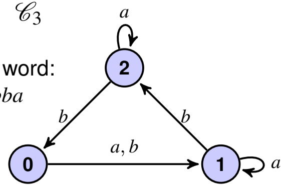
reset word:
abbbabbba



ČERNÝ'S INFINITE FAMILY

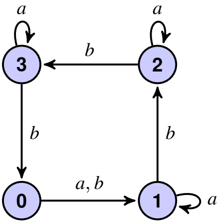
$\mathscr{C}_3$

reset word:
abba

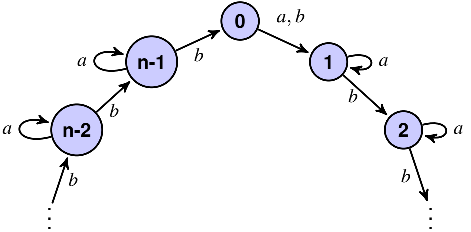


$\mathscr{C}_4$

reset word:
abbbabbba



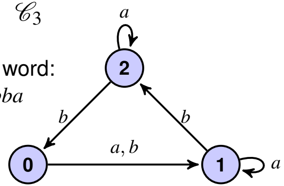
$\mathscr{C}_n$



ČERNÝ'S INFINITE FAMILY

$\mathscr{C}_3$

reset word: *abba*



$\mathscr{C}_4$

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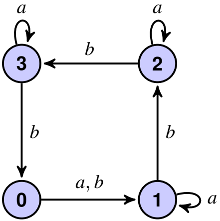

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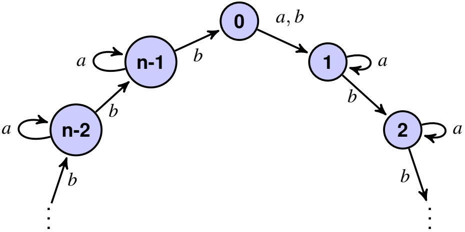
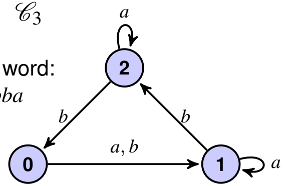


reset word: $abbb\cdots babbb\cdots ba = (ab^{n-1})^{n-2}a$

ČERNÝ'S INFINITE FAMILY

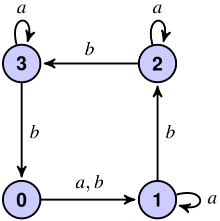
$\mathscr{C}_3$

reset word:
abba

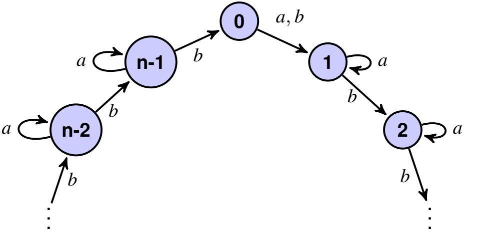


$\mathscr{C}_4$

reset word:
abbbabbba



$\mathscr{C}_n$



reset word: $abbb\cdots babbb\cdots ba = (ab^{n-1})^{n-2}a$

length: $n(n-2)+1 = (n-1)^2$

The Černý bound

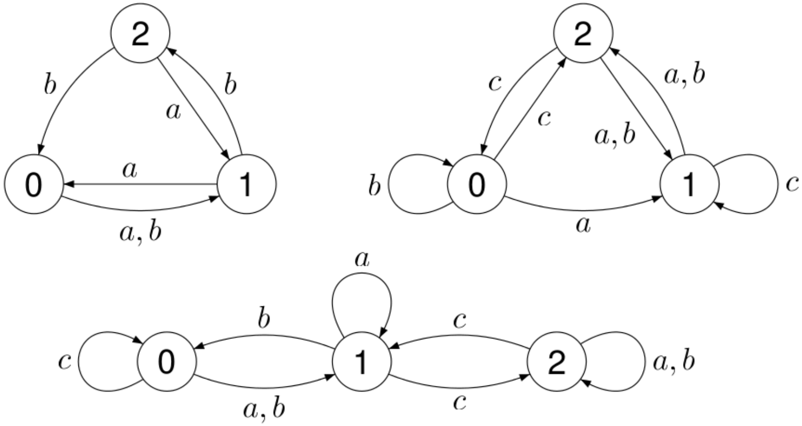
- A synchronizing automaton with n states *reaches the Černý bound* if the minimum length of all reset words is $(n-1)^2$.

- We present all known proper synchronizing automata with $n > 2$ states.

("proper" means removal of any letter results in a nonsynchronizing automaton)

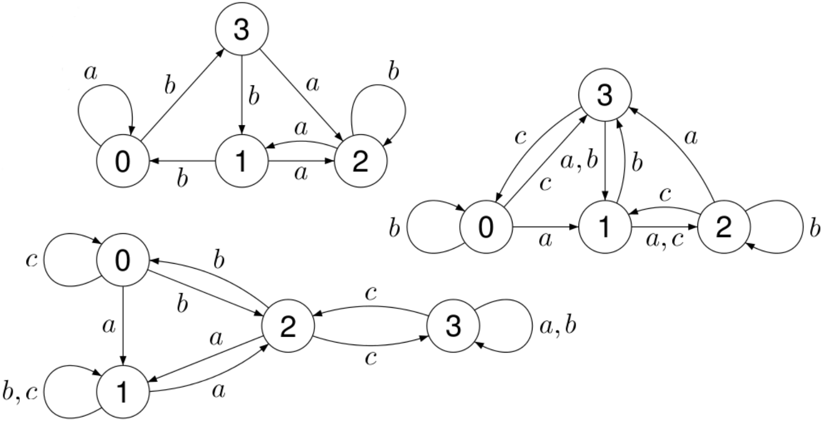
SPORADIC EXAMPLES

There are three sporadic examples on 3 states:



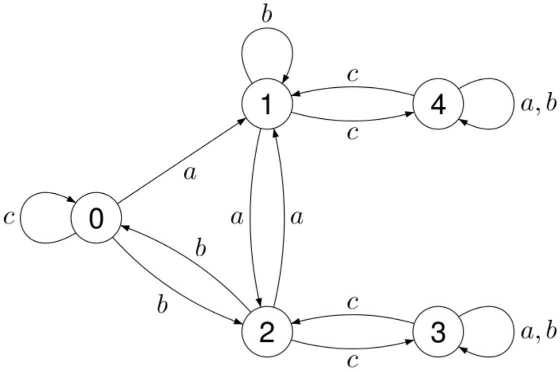
SPORADIC EXAMPLES

For 4 states, three sporadic examples are known:

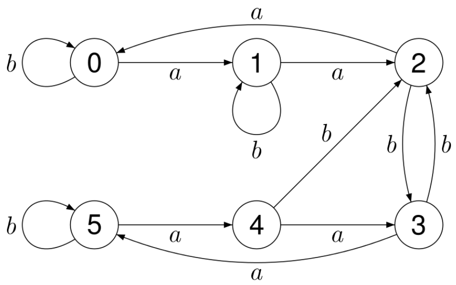


ROMAN'S EXAMPLE

For 5 states, a synchronizing automaton reaching the Černý bound was discovered by Adam Roman:



The last in our list and the most remarkable example was published in 2001 by Jarkko Kari.



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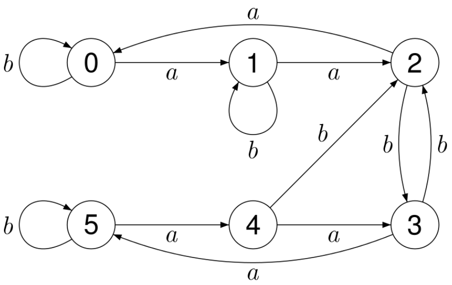

Minimum length reset word: $a\,b\,b\,a\,b\,a\,b\,b\,a\,b\,b\,a\,b\,a\,a\,b\,b\,a\,b\,a\,b\,a\,b\,b\,a$

CLONES

A BRIEF REVIEW

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$$p_k^n(x_1, \ldots, x_n) = x_k$$

- If $f \in A^{(A^n)}$ and $g_1, \ldots, g_n \in A^{(A^k)}$, then a *generalized composition* is

$$f[g_1, \ldots, g_n] : (x_1, \ldots, x_k) \mapsto f(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_k))$$

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- A *clone* on A is a subset of $\mathrm{Op}(A)$ that contains all projections and is closed under generalized compositions.
- Given $F \subseteq \mathrm{Op}(A)$, let $\mathrm{Clo}^A(F)$ denote the smallest clone on A containing F.
- Let $\mathrm{Clo}_n^A(F)$ denote the n-ary members of $\mathrm{Clo}^A(F)$.

Clones

A brief review

- If $\mathbf{A} = \langle A, F \rangle$ is an algebra, then $\mathrm{Clo}^A(F)$ is called the *clone of term operations on* $\mathbf{A}$, which we denote by $\mathrm{Clo}(\mathbf{A})$.

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Given $\mathbf{A}$ *and* $n \in \omega$*, the set* $\mathrm{Clo}_n(\mathbf{A})$ *is a subuniverse of the direct power algebra* $\mathbf{A}^{(A^n)}$*. It is generated by the projections* $\{p^n_1, p^n_2, \ldots, p^n_n\}$*.*

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EXAMPLE

$$A = \{0, 1, 2\} \qquad a = (1, 1, 2) \qquad b = (1, 2, 0) \qquad \mathbf{A} = \langle A, \{a, b\}\rangle$$

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Terms

A brief review

- Let $\mathcal{F}$ be a set of operation symbols and let $\rho : \mathcal{F} \to \omega$ be a similarity type.
- Let $F_n = \{f \in \mathcal{F} \mid \rho(f) = n\}$ be the set of n-ary operation symbols in $\mathcal{F}$.

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- Let X be a set, disjoint from $\mathcal{F}$, whose elements we call *variables*.
- A *word on the alphabet* $X \cup \mathcal{F}$ is a finite string $a_1 a_2 \cdots a_n$, where $a_i \in X \cup \mathcal{F}$. The *product* of words is defined by concatenation.

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- The set $T_\rho(X)$ of *terms of type* ρ *over* X is the smallest set T of words on the alphabet $X \cup \mathcal{F}$ such that
  - $X \cup F_0 \subseteq T$
  - If $t_1, \dots, t_n \in T$ and $f \in F_n$, then $f t_1 t_2 \cdots t_n \in T$.
- If $t_1, \dots, t_n$ are terms and $f \in F_n$ is an n-ary operation symbol, we often write $f(t_1, t_2, \cdots, t_n)$ instead of $f t_1 t_2 \cdots t_n$.

Terms

A brief review

- For each $f \in F_n$ let $f^{\mathbf{T}_\rho(X)}$ be the n-ary operation on $T_\rho(X)$ that maps $(t_1, \ldots, t_n)$ to $ft_1 \ldots t_n$.
- Define *term algebra of type* ρ as follows

$$\mathbf{T}_\rho(X) = \langle T_\rho(X), \{f^{\mathbf{T}_\rho(X)} : f \in \mathcal{F}\}\rangle$$

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- Let $X_n = \{x_1, \ldots, x_n\}$. For $t(x_1, \ldots, x_n) \in T_\rho(X_n)$ and $\mathbf{A}$ an algebra of type ρ, define an n-ary operation $t^{\mathbf{A}}$ on A by recursion on the "height" of t:

  – if t is the variable x_i then $t^{\mathbf{A}}(a_1, \ldots, a_n) = a_i$

  – if $t = fs_1 s_2 \ldots s_k$ where $f \in F_k$ and $s_1, \ldots, s_k$ are terms, then

$$t^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{A}}(s_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, s_k^{\mathbf{A}}(a_1, \ldots, a_n))$$

Connecting clones and terms

Consider the map $\phi : X_n \to A^{(A^n)}$ defined by $\phi(x_i) = p_i^n$.

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Theorem

$$\mathrm{Clo}_n(\mathbf{A}) = \{t^{\mathbf{A}} : t \in T_\rho(X_n)\}$$

Proof: For the unique hom above, we have $\bar{\phi}(t) = t^{\mathbf{A}} \in \mathrm{Clo}_n(\mathbf{A})$.

The image $\bar{\phi}(T_\rho(X_n))$ contains all the projections.

The projections generate the algebra $\mathbf{Clo}_n(\mathbf{A})$, so

$$\bar{\phi}(T_\rho(X_n)) = \mathrm{Clo}_n(\mathbf{A})$$

CONNECTING CLONES AND TERMS

...and finally, we recall that if $\mathbf{F}_{\mathbf{A}}(X_n)$ denotes a free algebra in $V(\mathbf{A})$ with free generating set X_n, then

$$\mathbf{F}_{\mathbf{A}}(X_n) \cong \mathbf{T}_{\rho}(X_n)/\Theta$$

where two terms are equivalent mod Θ when they induce the same term operation on every member of $V(\mathbf{A})$.

Thus, taking Θ to be the kernel of $\bar{\phi}$, we have $\mathbf{F}_{\mathbf{A}}(X_n) \cong \mathrm{Clo}_n(\mathbf{A})$.

Example

$$A = \{0, 1, 2\} \qquad a = (1, 1, 2) \qquad b = (1, 2, 0) \qquad \mathbf{A} = \langle A, \{a, b\}\rangle$$

$\mathrm{Clo}_1(\mathbf{A})$ is the subuniverse of $\mathbf{A}^A = \mathbf{A} \times \mathbf{A} \times \mathbf{A}$ generated by $(0, 1, 2)$

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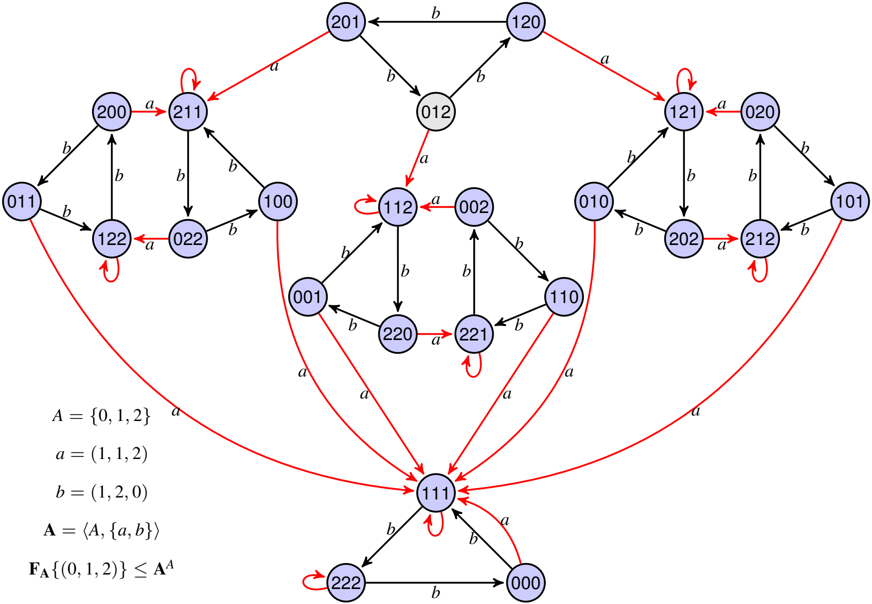

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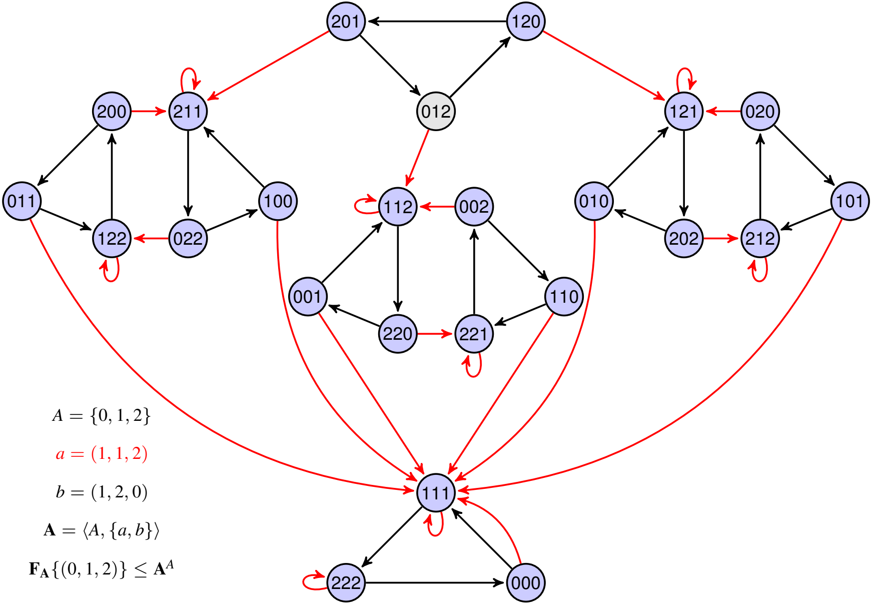

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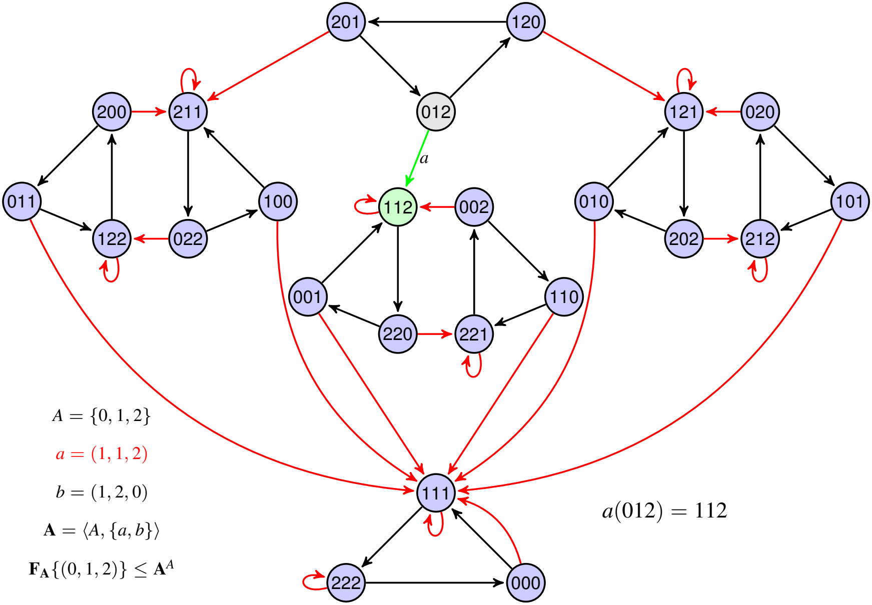
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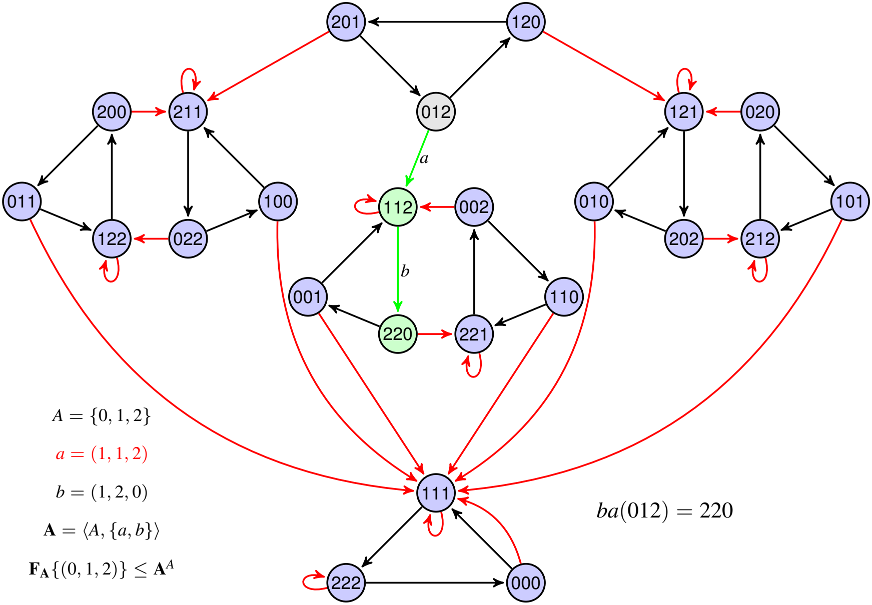
$a(012) = 112$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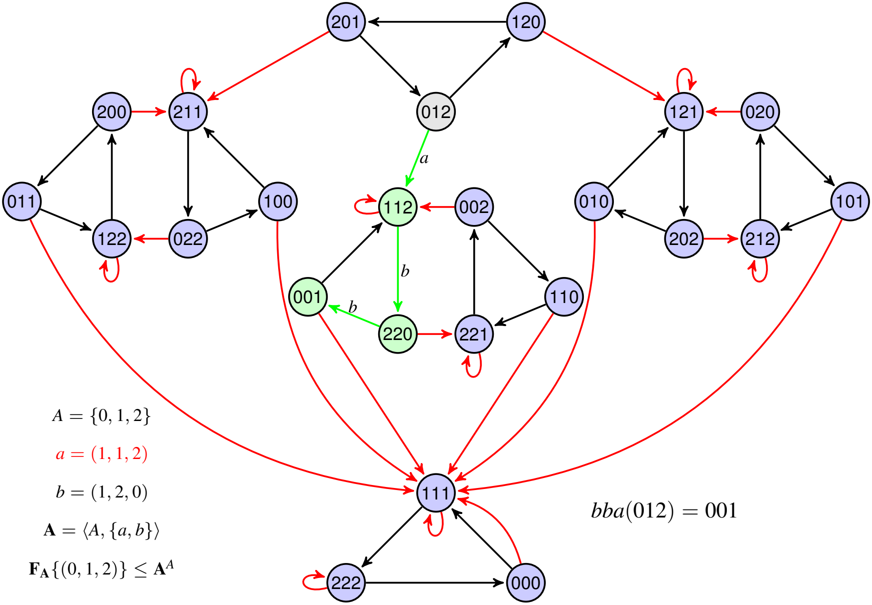
$ba(012) = 220$



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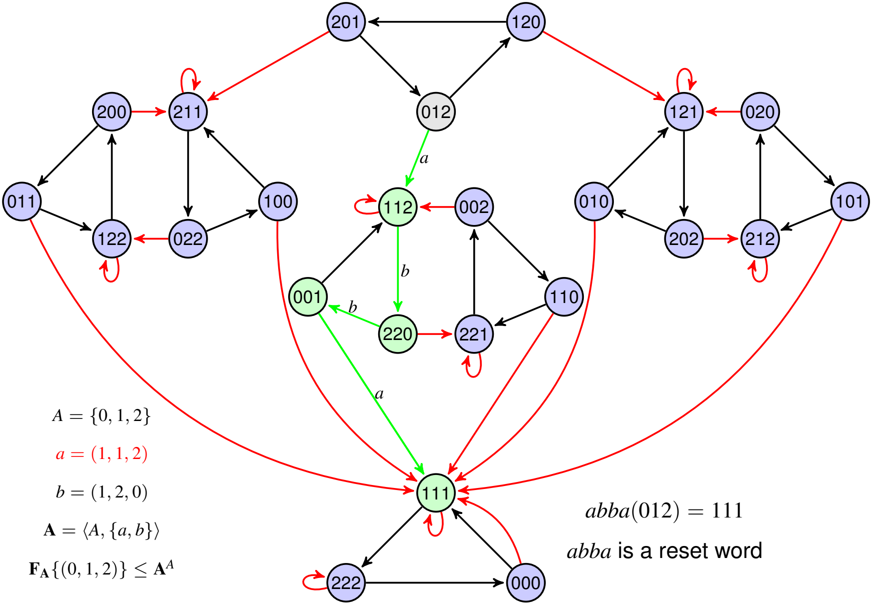
$bba(012) = 001$



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$\mathbf{F_A}\{(0, 1, 2)\} \leq \mathbf{A}^A$

$abba(012) = 111$

abba is a reset word